

# Black Hole Entropy and Finite Geometry

Péter Lévay,<sup>1</sup> Metod Saniga<sup>2</sup>, Péter Vrana<sup>1</sup> and Petr Pracna<sup>3</sup>

<sup>1</sup>Department of Theoretical Physics, Institute of Physics, Budapest University of Technology and Economics, H-1521 Budapest, Hungary

<sup>2</sup>Astronomical Institute, Slovak Academy of Sciences  
SK-05960 Tatranská Lomnica, Slovak Republic  
and

<sup>3</sup>J. Heyrovský Institute of Physical Chemistry, Academy of Sciences of the Czech Republic  
Dolejškova 3, CZ-182 23 Prague 8, Czech Republic

(9 March 2009)

---

**Abstract:** It is shown that the  $E_{6(6)}$  symmetric entropy formula describing black holes and black strings in  $D = 5$  is intimately tied to the geometry of the generalized quadrangle  $\text{GQ}(2, 4)$  with automorphism group the Weyl group  $W(E_6)$ . The 27 charges correspond to the points and the 45 terms in the entropy formula to the lines of  $\text{GQ}(2, 4)$ . Different truncations with 15, 11 and 9 charges are represented by three distinguished subconfigurations of  $\text{GQ}(2, 4)$ , well-known to finite geometers; these are the “doily” (i.e.  $\text{GQ}(2, 2)$ ) with 15, the “perp-set” of a point with 11, and the “grid” (i.e.  $\text{GQ}(2, 1)$ ) with 9 points, respectively. In order to obtain the correct signs for the terms in the entropy formula, we use a non-commutative labelling for the points of  $\text{GQ}(2, 4)$ . For the 40 different possible truncations with 9 charges this labelling yields 120 Mermin squares — objects well-known from studies concerning Bell-Kochen-Specker-like theorems. These results are connected to our previous ones obtained for the  $E_{7(7)}$  symmetric entropy formula in  $D = 4$  by observing that the structure of  $\text{GQ}(2, 4)$  is linked to a particular kind of geometric hyperplane of the split Cayley hexagon of order two, featuring 27 points located on 9 pairwise disjoint lines (a distance-3-spread). We conjecture that the different possibilities of describing the  $D = 5$  entropy formula using Jordan algebras, qubits and/or qutrits correspond to employing different coordinates for an underlying non-commutative geometric structure based on  $\text{GQ}(2, 4)$ .

---

## 1 Introduction

Recently striking multiple relations have been established between the physics of stringy black hole solutions and quantum information theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Though this “black hole analogy” still begs for a firm physical basis, the underlying correspondences have repeatedly proved to be useful for obtaining new insights into one of these fields by exploiting the methods established within the other. The main unifying theme in these papers is the correspondence between the Bekenstein-Hawking entropy formula [13, 14] for black-hole and black-string solutions in  $D = 4$  and  $D = 5$  supergravities arising from string/M-theory compactifications and certain entanglement invariants of multi-qubit/-qutrit systems. As a new unifying agent in some of these papers [5, 12] the role of discrete geometric ideas have been emphasized. In particular it has been shown [4, 5] that the Fano plane with seven points and seven lines with points conveniently labelled by seven three-qubit states can be used to describe the structure of the  $E_{7(7)}$  symmetric black hole entropy formula of  $N = 8$ ,  $D = 4$  supergravity. Moreover, this geometric representation based on the fundamental 56 dimensional representation of  $E_{7(7)}$  in terms of 28 electric and 28 magnetic charges enabled a diagrammatic understanding of the consistent truncations with 32, 24 and 8 charges as a restriction to quadrangles, lines and points of the Fano plane [5]. Though the Fano plane turned out to be a crucial ingredient also in later studies, this geometric representation based on the tripartite entanglement of seven qubits has a number of shortcomings [10]. In order to eliminate these, in our latest paper [12] we attempted to construct a new representation using merely *three-qubits*. The basic idea was to use the

central quotient of the three-qubit Pauli group [15], well-known from studies concerning quantum error correcting codes. This Abelian group can be described by the 63 *real* operators of the Pauli group with multiplication up to a sign. These 63 operators can be mapped bijectively to the points of a finite geometrical object called the split Cayley hexagon of order two having 63 points and 63 lines, with a subgeometry (the complement of one of its geometric hyperplanes) being the Coxeter graph with 28 points/vertices. This graph has been related to the charge configurations of the  $E_7$  symmetric black hole entropy formula [12]. The advantage of this representation was a clear understanding of an automorphism of order seven relating the *seven* STU subsectors of  $N = 8$ ,  $D = 4$  supergravity and the explicit appearance of a discrete  $PSL(2, 7)$  symmetry of the black hole entropy formula. The permutation symmetry of the STU model (triality) in this picture arises as a subgroup of  $PSL(2, 7)$ .

Encouraged by the partial success of finite geometric ideas in the  $D = 4$  case the aim of the present paper is to shed some light on a beautiful finite geometric structure underlying also the  $E_{6(6)}$  symmetric entropy formula in  $D = 5$ . We show that in this case the relevant finite geometric objects are *generalized quadrangles* with lines of size three. As it is well-known black holes in  $D = 5$  have already played a special role in string theory, since these objects provided the first clue how to understand the microscopic origin of the Bekenstein-Hawking entropy [17].

As a first step, in Section 2 we emphasize that according to several well-known theorems [16, 18, 19] we have just *four* (including also a “weak/degenerate” one made of all lines passing through a fixed point [20]) such quadrangles, which are directly related to the *four* possible division algebras. It is well-known that magic  $N = 2$ ,  $D = 5$  supergravities [21, 22, 23, 10] coupled to 5, 8, 14 and 26 vector multiplets with symmetries  $SL(3, \mathbf{R})$ ,  $SL(3, \mathbf{C})$ ,  $SU^*(6)$  and  $E_{6(-26)}$  can be described by Jordan algebras of  $3 \times 3$  Hermitian matrices with entries taken from the real and complex numbers, quaternions and octonions. It is also known that in these cases we have black hole solutions that have cubic invariants whose square roots yield the corresponding black hole entropy [24]. Moreover, we can also replace in these Jordan algebras the division algebras by their split versions. For example, in this way in the case of split octonions we arrive at the  $N = 8$ ,  $D = 5$  supergravity [25] with 27 Abelian gauge fields transforming in the fundamental of  $E_{6(6)}$ . In this theory the corresponding black hole solutions have an entropy formula having  $E_{6(6)}(\mathbf{Z})$  symmetry [26, 27, 10]. This analogy existing between division algebras, Jordan algebras and generalized quadrangles with lines having three points leads us to a conjecture that such finite geometric objects should be relevant for a fuller geometrical understanding of black hole entropy in  $D = 5$ .

In Section 3, by establishing an explicit mapping between the 27 points and 45 lines of the generalized quadrangle  $GQ(2, 4)$  and the 27 charges and the 45 terms in the cubic invariant appearing in the entropy formula, we prove that our conjecture is true. The crucial observation here is that the automorphism group of  $GQ(2, 4)$  is the Weyl group  $W(E_6)$  with order 51840. Our labelling for the points of  $GQ(2, 4)$  used here is a one directly related to the two qutrit states of Duff and Ferrara [7]. By using the vocabulary of Borsten et al. [10], this labelling directly relates to the usual one featuring cubic Jordan algebras.

In Section 4 we observe that our geometric correspondence merely gives the number and structure of the terms in the cubic invariant. In the case of the  $E_{6(6)}(\mathbf{Z})$  symmetric black hole entropy in order to produce also the correct signs of these terms we have to employ a noncommutative labelling for the points of  $GQ(2, 4)$ . To link these considerations to our previous paper on the  $E_{7(7)}(\mathbf{Z})$  symmetric black hole entropy in  $D = 4$ , we adopt the labelling by real three-qubit operators of the Pauli group. We show that this labelling scheme is connected to a certain type of geometric hyperplane of the split Cayley hexagon of order two featuring precisely 27 points that lie on 9 pairwise disjoint lines. There are 28 different hyperplanes of this kind in the hexagon, giving rise to further possible labellings. Next, we focus on special subconfigurations of  $GQ(2, 4)$  which are called grids. These are generalized quadrangles  $GQ(2, 1)$ , featuring 9 points and 6 lines that can be arranged in the form of squares. There are 120 distinct copies of them living within  $GQ(2, 4)$ , grouped to 40 triples such that each of them comprises all of the 27 points of  $GQ(2, 4)$ . Our noncommutative labelling renders these grids to Mermin squares, which are objects of great relevance for obtaining very economical proofs to Bell-Kochen-Specker-like theorems. In order to complete

the paper, we also present the action of the Weyl group on the noncommutative labels of  $GQ(2, 4)$ . This also provides a proof for the  $W(E_6)$  invariance of the cubic invariant.

Finally, Section 5 highlights our main findings and presents our conclusive remarks and conjectures. In particular, we conjecture that the different possibilities of describing the  $D = 5$  entropy formula using Jordan algebras, qubits and/or qutrits correspond to employing different coordinates for an underlying noncommutative geometric structure based on  $GQ(2, 4)$ .

## 2 Jordan algebras and generalized quadrangles

### 2.1 Cubic Jordan algebras

As we remarked in the introduction the charge configurations of  $D = 5$  black holes/strings are related to the structure of cubic Jordan algebras. An element of a cubic Jordan algebra can be represented as a  $3 \times 3$  Hermitian matrix with entries taken from a division algebra  $\mathbf{A}$ , i. e.  $\mathbf{R}$ ,  $\mathbf{C}$ ,  $\mathbf{H}$  or  $\mathbf{O}$ . (The real and complex numbers, the quaternions and the octonions.) Explicitly, we have

$$J_3(Q) = \begin{pmatrix} q_1 & Q^v & \overline{Q^s} \\ \overline{Q^v} & q_2 & Q^c \\ Q^s & \overline{Q^c} & q_3 \end{pmatrix} \quad q_i \in \mathbf{R}, \quad Q^{v,s,c} \in \mathbf{A}, \quad (1)$$

where an overbar refers to conjugation in  $\mathbf{A}$ . These charge configurations describe electric black holes of the  $N = 2$ ,  $D = 5$  magic supergravities [21, 22, 23, 10]. In the octonionic case the superscripts of  $Q$  refer to the fact that the fundamental 27 dimensional representation of the U-duality group  $E_{6(-26)}$  decomposes under the subgroup  $SO(8)$  to three 8 dimensional representations (vector, spinor and conjugate spinor) connected by triality and to three singlets corresponding to the  $q_i, i = 1, 2, 3$ . Note that a general element in this case is of the form  $Q = Q_0 + Q_1 e_1 + \dots + Q_7 e_7$ , where the “imaginary units”  $e_1, e_2, \dots, e_7$  satisfy the rules of the octonionic multiplication table [10]. The norm of an octonion is  $Q\overline{Q} = (Q_0)^2 + \dots + (Q_7)^2$ . The real part of an octonion is defined as  $\text{Re}(Q) = \frac{1}{2}(Q + \overline{Q})$ .

The magnetic analogue of  $J_3(Q)$  is

$$J_3(P) = \begin{pmatrix} p^1 & P^v & \overline{P^s} \\ \overline{P^v} & p^2 & P^c \\ P^s & \overline{P^c} & p^3 \end{pmatrix} \quad p^i \in \mathbf{R}, \quad P^{v,s,c} \in \mathbf{A}, \quad (2)$$

describing black strings related to the previous case by the electric-magnetic duality. The black hole entropy is given by the cubic invariant

$$I_3(Q) = q_1 q_2 q_3 - (q_1 Q^s \overline{Q^s} + q_2 Q^c \overline{Q^c} + q_3 Q^v \overline{Q^v}) + 2\text{Re}(Q^v Q^s Q^c), \quad (3)$$

as

$$S = \pi \sqrt{I_3(Q)}, \quad (4)$$

and for the black string we get a similar formula with  $I_3(Q)$  replaced by  $I_3(P)$ . Recall that  $I_3$  is just the norm of the cubic Jordan algebra and the norm preserving group is  $SL(3, \mathbf{A})$  and  $J_3^{\mathbf{A}}$  transforms under this group with respect to the  $3\dim\mathbf{A} + 3$  dimensional representation, i. e. as the **6**, **9**, **15** and **27** of the groups  $SL(3, \mathbf{R})$ ,  $SL(3, \mathbf{C})$ ,  $SU^*(6)$  and  $E_{6(-26)}$ .

We can also consider cubic Jordan algebras with  $\mathbf{C}$ ,  $\mathbf{H}$  and  $\mathbf{O}$  replaced by the corresponding *split versions*. In the octonionic case  $\mathbf{O}_s$  the “norm” is defined as

$$Q\overline{Q} = (Q_0)^2 + (Q_1)^2 + (Q_2)^2 + (Q_3)^2 - (Q_4)^2 - (Q_5)^2 - (Q_6)^2 - (Q_7)^2, \quad (5)$$

and the group preserving the norm of the corresponding Jordan algebra is  $E_{6(6)}$ , which decomposes similarly under  $SO(4, 4)$ . This is the case of  $N = 8$  supergravity with duality

group  $E_{6(6)}$  [28]. Note that the groups  $E_{6(-26)}$  and  $E_{6(6)}$  are the symmetry groups of the corresponding classical supergravity. In the quantum theory the black hole/string charges become integer-valued and the relevant  $3 \times 3$  matrices are defined over the *integral* octonions and *integral* split octonions, respectively. Hence, the U-duality groups are in this case broken to  $E_{6(-26)}(\mathbf{Z})$  and  $E_{6(6)}(\mathbf{Z})$  accordingly. In all these cases the entropy formula is given by Eqs. (3)–(4), with the norm given by either the usual one or its split analogue, Eq. (5).

It is also important to recall that the magic  $N = 2$  supergravities associated with the real and complex numbers and the quaternions can be obtained as consistent reductions of the  $N = 8$  one [24] which is based on the *split octonions*. On the other hand, the  $N = 2$  supergravity based on the division algebra of the octonions is exceptional since it is the only one that cannot be obtained from the split octonionic  $N = 8$  one by truncation.

## 2.2 Finite generalized quadrangles

Now we summarize the basic definitions on generalized quadrangles that we'll need later. A *finite generalized quadrangle* of order  $(s, t)$ , usually denoted  $\text{GQ}(s, t)$ , is an incidence structure  $S = (P, B, I)$ , where  $P$  and  $B$  are disjoint (non-empty) sets of objects, called respectively points and lines, and where  $I$  is a symmetric point-line incidence relation satisfying the following axioms [29]: (i) each point is incident with  $1 + t$  lines ( $t \geq 1$ ) and two distinct points are incident with at most one line; (ii) each line is incident with  $1 + s$  points ( $s \geq 1$ ) and two distinct lines are incident with at most one point; and (iii) if  $x$  is a point and  $L$  is a line not incident with  $x$ , then there exists a unique pair  $(y, M) \in P \times B$  for which  $xIMyLL$ ; from these axioms it readily follows that  $|P| = (s + 1)(st + 1)$  and  $|B| = (t + 1)(st + 1)$ . It is obvious that there exists a point-line duality with respect to which each of the axioms is self-dual. Interchanging points and lines in  $S$  thus yields a generalized quadrangle  $S^D$  of order  $(t, s)$ , called the dual of  $S$ . If  $s = t$ ,  $S$  is said to have order  $s$ . The generalized quadrangle of order  $(s, 1)$  is called a grid and that of order  $(1, t)$  a dual grid. A generalized quadrangle with both  $s > 1$  and  $t > 1$  is called thick. In any  $\text{GQ}(s, t)$ ,  $s + t$  divides both  $st(1 + st)$  [16] and  $st(s + 1)(t + 1)$  [18]; moreover, if  $s > 1$  (dually,  $t > 1$ ) then  $t \leq s^2$  (dually,  $s \leq t^2$ ) [19].

Given two points  $x$  and  $y$  of  $S$  one writes  $x \sim y$  and says that  $x$  and  $y$  are collinear if there exists a line  $L$  of  $S$  incident with both. For any  $x \in P$  denote  $x^\perp = \{y \in P | y \sim x\}$  and note that  $x \in x^\perp$ ; obviously,  $|x^\perp| = 1 + s + st$ . Given an arbitrary subset  $A$  of  $P$ , the *perp*-(set) of  $A$ ,  $A^\perp$ , is defined as  $A^\perp = \bigcap \{x^\perp | x \in A\}$  and  $A^{\perp\perp} := (A^\perp)^\perp$ . An ovoid of a generalized quadrangle  $S$  is a set of points of  $S$  such that each line of  $S$  is incident with exactly one point of the set; hence, each ovoid contains  $st + 1$  points.

A *geometric hyperplane*  $H$  of a point-line geometry  $\Gamma(P, B)$  is a proper subset of  $P$  such that each line of  $\Gamma$  meets  $H$  in one or all points [30]. For  $\Gamma = \text{GQ}(s, t)$ , it is well known that  $H$  is one of the following three kinds: (i) the *perp*-set of a point  $x$ ,  $x^\perp$ ; (ii) a (full) subquadrangle of order  $(s, t')$ ,  $t' < t$ ; and (iii) an ovoid.

In what follows, we shall be uniquely concerned with generalized quadrangles having lines of size *three*,  $\text{GQ}(2, t)$ . From the above-given restrictions on parameters of  $\text{GQ}(s, t)$  one readily sees that these are of three distinct kinds, namely  $\text{GQ}(2, 1)$ ,  $\text{GQ}(2, 2)$  and  $\text{GQ}(2, 4)$ , each unique. They can uniformly be characterized as being formed by the points and lines of a hyperbolic, a parabolic and an elliptic quadric in three-, four- and five-dimensional projective space over  $\text{GF}(2)$ , respectively.  $\text{GQ}(2, 1)$  is a grid of 9 points on 6 lines, being the complement of the lattice graph  $K_3 \times K_3$ . It contains only ovoids (6; each of size 3) and *perp*-sets (9; each of size 5).  $\text{GQ}(2, 1)$  is obviously different from its dual, the complete bipartite graph on 6 vertices.  $\text{GQ}(2, 2)$  is the smallest thick generalized quadrangle, also known as the “doily.” This quadrangle is endowed with 15 points/lines, with each line containing 3 points and, dually, each point being on 3 lines; moreover, it is a self-dual object, i. e., isomorphic to its dual. It is the complement of the triangular graph  $T(6)$  and features all the three kinds of geometric hyperplanes, of the following cardinalities: 15 *perp*-sets,  $x^\perp$ , 7 points each; 10 grids (i. e.  $\text{GQ}(2, 1)$ s), 9 points each; and 6 ovoids, 5 points each. One of its most familiar constructions is in terms of synthemes and duads, where the point set consists of all pairs of a six-element set and the line set comprises all three-sets of pairs forming a partition of the six-element set. The full group of automorphisms of  $\text{GQ}(2, 2)$  is  $S_6$ , of order

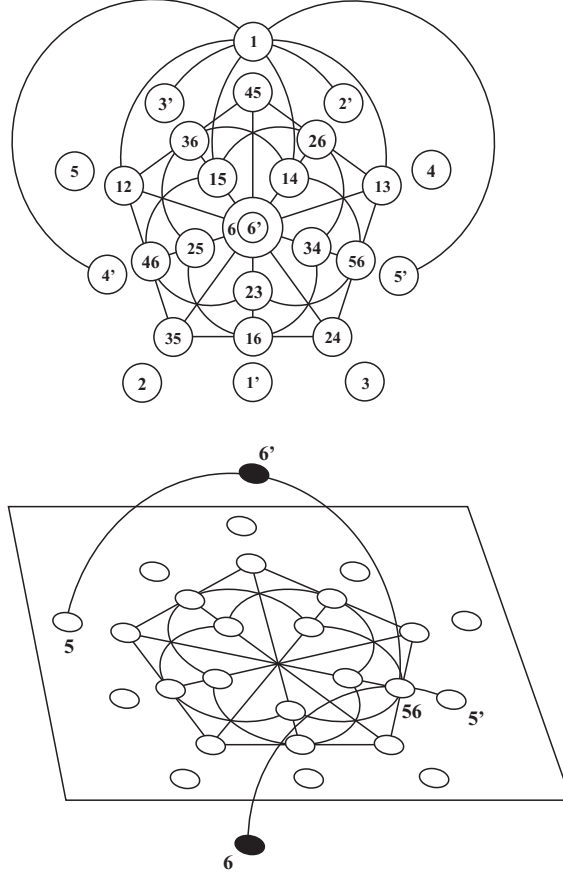


Figure 1: A diagrammatic illustration of the structure of the generalized quadrangle  $GQ(2, 4)$  after Polster [31]. In both the figures, each picture depicts all 27 points (circles). The top picture shows only 19 lines (line segments and arcs of circles) of  $GQ(2, 4)$ , with the two points located in the middle of the doily being regarded as lying one above and the other below the plane the doily is drawn in. 16 out of the missing 26 lines can be obtained by successive rotations of the figure through 72 degrees around the center of the pentagon. The bottom picture shows a couple of lines which go off the doily's plane; the remaining 8 lines of this kind are again got by rotating the figure through 72 degrees around the center of the pentagon.

720. The last case in the hierarchy is  $GQ(2, 4)$ , which possesses 27 points and 45 lines, with lines of size 3 and 5 lines through a point. Its full group of automorphisms is of order 51840, being isomorphic to the Weyl group  $W(E_6)$ .  $GQ(2, 4)$  is obviously not a self-dual structure; its dual,  $GQ(4, 2)$ , features 45 points and 27 lines, with lines of size 5 and 3 lines through a point. Unlike its dual, which exhibits all the three kinds of geometric hyperplanes,  $GQ(2, 4)$  is endowed only with perp-sets (27, of cardinality 11 each) and  $GQ(2, 2)$ s (36), *not* admitting any ovoid. One of its constructions goes as follows. One starts with the above-introduced syntheme-duad construction of  $GQ(2, 2)$ , adds 12 more points labelled simply as 1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6' and defines 30 additional lines as the three-sets  $\{a, b', \{a, b\}\}$  of points, where  $a, b \in \{1, 2, 3, 4, 5, 6\}$  and  $a \neq b$  — as diagrammatically illustrated, after Polster [31], in Figure 1. To conclude this section, we emphasize the fact that  $GQ(2, 1)$  is a geometric hyperplane of  $GQ(2, 2)$ , which itself is a geometric hyperplane of  $GQ(2, 4)$ .

Looking at the sequence of numbers 27, 15 and 9, representing the number of points of these quadrangles, one is immediately tempted to relate these numbers to the dimensions of the representations of the norm-preserving groups of the cubic Jordan algebras based on  $\mathbf{O}$  (or  $\mathbf{O}_s$ ),  $\mathbf{H}$  and  $\mathbf{C}$ . After a quick glance at the structure of the corresponding entropy

formulas [24] constructed within the context of magic supergravities one also recognizes that the sequence 45, 15 and 6, representing the number of lines, should correspond to the number of different terms in the corresponding entropy formulas. (Thus, for example, 15 is the number of terms in the Pfaffian of a  $6 \times 6$  antisymmetric matrix giving rise to the quaternionic magic entropy formula. Moreover, 6 is the number of terms in the determinant of a  $3 \times 3$  matrix occurring in the entropy formula of the complex case.) Indeed, using the nice labelling scheme developed by Duff and his coworkers[10] it is not difficult to set up an explicit geometric correspondence between the 45 lines of  $\text{GQ}(2, 4)$  and the terms in the cubic invariant of Eq. (3). This is the task we now turn to.

### 3 The cubic invariant and GQs with lines of size three

#### 3.1 $\text{GQ}(2, 4)$ and qutrits

Since except for the octonionic magic all the  $N = 2$  magic supergravities can be obtained as consistent truncations of the  $N = 8$  split-octonionic case, let us consider the cubic invariant  $I_3$  of Eq. (3) with the  $U$ -duality group  $E_{6(6)}$ . Let us consider the decomposition of the 27 dimensional fundamental representation of  $E_{6(6)}$  with respect to its  $SL(3, \mathbf{R})^{\otimes 3}$  subgroup. We have the decomposition

$$E_{6(6)} \supset SL(3, \mathbf{R})_A \times SL(3, \mathbf{R})_B \times SL(3, \mathbf{R})_C \quad (6)$$

under which

$$\mathbf{27} \rightarrow (\mathbf{3}', \mathbf{3}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{3}', \mathbf{3}') \otimes (\mathbf{3}, \mathbf{1}, \mathbf{3}). \quad (7)$$

As it is known [7, 10], the above-given decomposition is related to the ‘‘bipartite entanglement of three-qutrits’’ interpretation of the  $\mathbf{27}$  of  $E_6(\mathbf{C})$ . Neglecting the details, all we need is three  $3 \times 3$  real matrices  $a, b$  and  $c$  with the index structure

$$a^A_B, \quad b^{BC}, \quad c_{CA}, \quad A, B, C = 0, 1, 2, \quad (8)$$

where the upper indices are transformed according to the (contragredient)  $\mathbf{3}'$  and the lower ones by  $\mathbf{3}$ . Then according to the dictionary developed in Borsten et al. [10], we have

$$p^1 = -a^0_0, \quad p^2 = -a^1_1, \quad p^3 = -a^3_3, \quad (9)$$

$$\begin{aligned} 2P^c = & - (a^1_2 + a^2_1)e_0 - (b^{00} + c_{00})e_1 - (b^{01} + c_{10})e_2 - (b^{02} + c_{20})e_3 \\ & + (a^1_2 - a^2_1)e_4 + (b^{00} - c_{00})e_5 + (b^{01} - c_{10})e_6 + (b^{02} - c_{20})e_7, \end{aligned} \quad (10)$$

$$\begin{aligned} 2P^s = & - (a^2_0 + a^0_2)e_0 - (b^{10} + c_{01})e_1 - (b^{11} + c_{11})e_2 - (b^{12} + c_{21})e_3 \\ & + (a^2_0 - a^0_2)e_4 + (b^{10} - c_{01})e_5 + (b^{11} - c_{11})e_6 + (b^{12} - c_{21})e_7, \end{aligned} \quad (11)$$

$$\begin{aligned} 2P^v = & - (a^0_1 + a^1_0)e_0 - (b^{20} + c_{02})e_1 - (b^{21} + c_{12})e_2 - (b^{22} + c_{22})e_3 \\ & + (a^0_1 - a^1_0)e_4 + (b^{20} - c_{02})e_5 + (b^{21} - c_{12})e_6 + (b^{22} - c_{22})e_7. \end{aligned} \quad (12)$$

We can express  $I_3$  of Eq. (3) in the alternative form as

$$I_3 = \text{Det} J_3(P) = a^3 + b^3 + c^3 + 6abc. \quad (13)$$

Here

$$a^3 = \frac{1}{6} \varepsilon_{A_1 A_2 A_3} \varepsilon^{B_1 B_2 B_3} a^{A_1}_{B_1} a^{A_2}_{B_2} a^{A_3}_{B_3}, \quad (14)$$

$$b^3 = \frac{1}{6} \varepsilon_{B_1 B_2 B_3} \varepsilon_{C_1 C_2 C_3} b^{B_1 C_1} b^{B_2 C_2} b^{B_3 C_3}, \quad (15)$$

$$c^3 = \frac{1}{6} \varepsilon^{C_1 C_2 C_3} \varepsilon^{A_1 A_2 A_3} c_{C_1 A_1} c_{C_2 A_2} c_{C_3 A_3}, \quad (16)$$

$$abc = \frac{1}{6} a^A{}_B b^{BC} c_{CA}. \quad (17)$$

Notice that the terms like  $c^3$  produce just the determinant of the corresponding  $3 \times 3$  matrix. Since each determinant contributes 6 terms, altogether we have 18 terms from the first three terms in Eq. (13). Moreover, since it is easy to see that the fourth term contains 27 terms, altogether  $I_3$  contains precisely 45 terms, i. e. the number which is equal to that of lines in  $\text{GQ}(2, 4)$ .

In order to set up a bijection between the points of  $\text{GQ}(2, 4)$  and the 27 amplitudes of the two-qutrit states of Eq. (8), we use the basic ideas of the above-given construction of  $\text{GQ}(2, 4)$  (see Figure 1). Since the automorphism group of the doily ( $\text{GQ}(2, 2)$ ) is the symmetric group  $S_6$ , this construction is based on labelling the 15 points of the doily by the 15 two-element subsets of the set  $\{1, 2, 3, 4, 5, 6\}$  on which  $S_6$  acts naturally. The next step consists of adding two six element sets: the basic set  $\{1, 2, 3, 4, 5, 6\}$  and an extra one  $\{1', 2', 3', 4', 5', 6'\}$  according to the rule as explained in Figure 1. Hence, the duad labelling is:  $(ij), i < j, (i)$  and  $(j')$  where  $i, j = 1, 2, \dots, 6$ .

We can easily relate the labelling of these 27 points to the structure of two  $8 \times 8$  antisymmetric matrices with 28 independent components, each with one special component removed. Let us label the rows and columns of such a matrix by the letters  $I, J = 0, 1, 2, \dots, 7$ . (The reason for this unusual labelling will be clarified in the next section.) Let us choose the special component to be removed from both of our  $8 \times 8$  matrices to be the element 01. Then we choose a  $6 \times 6$  antisymmetric block in one of the  $8 \times 8$  matrices labelled by the row and column indices  $I < J, I, J = 2, 3, \dots, 7$ . Its 15 components give rise to a duad labelling of the doily. Now, from the other  $8 \times 8$  matrix we choose the elements of the form  $0J$  with  $J = 2, \dots, 7$  to correspond to the set  $\{1', \dots, 6'\}$ , and the ones of the form  $1J$  to the one  $\{1, \dots, 6\}$ . (Clearly, the row and column indices are shifted by one unit with respect to the usual duad indices, i. e.  $I = i + 1, J = j + 1$ .) Now, it is well-known that the cubic  $E_{6(6)}$  invariant for  $D = 5$  black hole solutions is related to the quartic  $E_{7(7)}$  invariant for  $D = 4$  ones by a suitable truncation of the Freudenthal triple system to the corresponding cubic Jordan algebra [32]. Recently, the Freudenthal triple description of the  $D = 4$  black hole entropy was related to the usual description due to Cartan using two  $8 \times 8$  antisymmetric matrices [10], corresponding to the 28 electric and 28 magnetic charges. Using Table 32 of Ref. 10, giving a dictionary between these descriptions, it is easy to realize that the 27 elements of the cubic Jordan algebra  $J_3(P)$  split as  $27 = 15 + 12$  between these two  $8 \times 8$  matrices. This automatically defines a one-to-one mapping between the duad construction of  $\text{GQ}(2, 4)$  and the 27 elements of  $J_3(P)$ . As the last step, using Eqs. (9–12) we can readily relate the arising  $J_3(P)$  labelling of  $\text{GQ}(2, 4)$  to the one in terms of two-qutrit amplitudes of Eq. (8). The explicit relationship between the duad labelling and the qutrit one is as follows

$$\{1, 2, 3, 4, 5, 6\} = \{c_{21}, a^2{}_1, b^{01}, a^0{}_1, c_{01}, b^{21}\}, \quad (18)$$

$$\{1', 2', 3', 4', 5', 6'\} = \{b^{10}, c_{10}, a^1{}_2, c_{12}, b^{12}, a^1{}_0\}, \quad (19)$$

$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\} = \{c_{02}, b^{22}, c_{00}, a^1{}_1, b^{02}, a^0{}_0, b^{11}, c_{22}, a^0{}_2\}, \quad (20)$$

$$\{34, 35, 36, 45, 46, 56\} = \{a^2{}_0, b^{20}, c_{11}, c_{20}, a^2{}_2, b^{00}\}. \quad (21)$$

This relationship is easily grasped by comparing Figure 2, which depicts the qutrit labelling, with Figure 1 (top).

Next, notice that the lines of  $\text{GQ}(2, 4)$  are of two types. They are either of the form  $(i, ij, j')$  or  $(ij, kl, mn)$ , where  $i, j, j', \dots = 1, \dots, 6$  and  $i, j, k, l, m, n$  are different. We have 30 lines of the first and 15 lines of the second type. The latter ones belong to the doily. Notice also that the three two-qutrit states of Eq. (8) partition the 27 points of  $\text{GQ}(2, 4)$  to

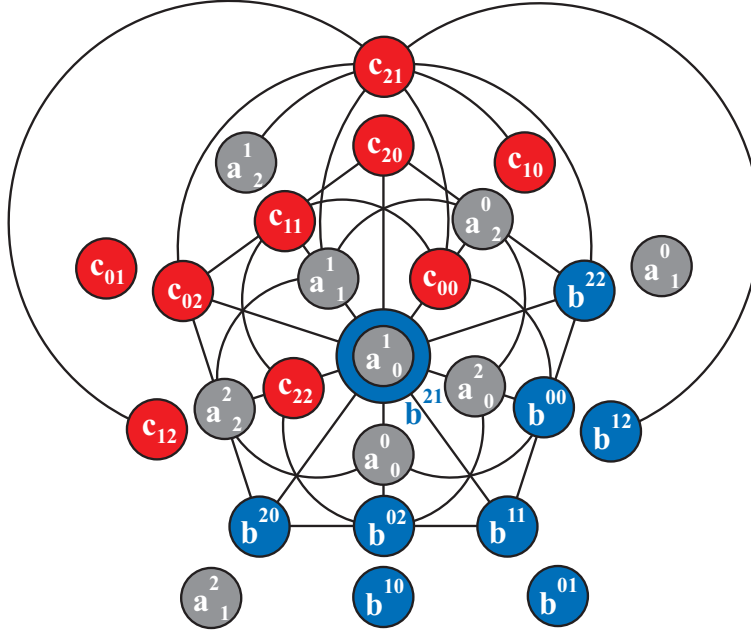


Figure 2: A qutrit labelling of the points of  $GQ(2, 4)$ . Three different colours (online only) are used to illustrate a triple of grids partitioning the point set.

3 disjoint grids, i.e.  $GQ(2, 1)$ s. The points of these three grids are coloured differently (in an online version only). The 27 lines corresponding to the terms of  $\text{Tr}(abc)$  of Eq. (13) are of the type like the one  $a^1_2 b^{22} c_{21}$ , and the  $3 \times 6 = 18$  terms are coming from the three  $3 \times 3$  determinants  $a^3, b^3, c^3$ . These terms are of the form like the one  $b^{20} b^{02} b^{11}$ . From Figure 2 one can check that each of 45 lines of  $GQ(2, 4)$  corresponds to exactly one monomial of Eq. (13).

We close this subsection with an important comment/observation. It is well-known that the automorphism group of the generalized quadrangle  $GQ(2, 4)$  is the Weyl group [29]  $W(E_6)$  of order 51840. Moreover, the cubic invariant is also connected to the geometry of smooth (non-singular) cubic surfaces in  $\mathbf{CP}^3$ . It is a classical result that the automorphism group of the configuration of 27 lines on a cubic can also be identified with  $W(E_6)$ . It is also known that different configurations of lines are related to special models of exceptional Lie algebras [33]. Indeed, it was Elie Cartan who first realized [34] that the 45 monomials of our cubic form stabilized by  $E_6$  are in correspondence with the tritangent planes of the cubic. In the light of this fact, our success in parametrizing the monomials of  $I_3$  using the lines of  $GQ(2, 4)$  is not at all surprising.

### 3.2 Geometric hyperplanes and truncations

Let us focus now on geometric hyperplanes of  $GQ(2, 4)$ . As already mentioned in Sec. 2.2, the only type of hyperplanes featured by  $GQ(2, 4)$  are doilies (36) and perp sets (27). Moreover,  $GQ(2, 4)$  also contains  $3 \times 40 = 120$  grids; however, these are *not* its geometric hyperplanes [35]. (This is quite different from the  $GQ(2, 2)$  case, where grids are geometric hyperplanes.) Though they are not hyperplanes, they have an important property that there exists 40 triples of them, each partitioning the point set of  $GQ(2, 4)$ .

It is easy to find a physical interpretation of the hyperplanes of  $GQ(2, 4)$ . The doily has 15 lines, hence we should have a truncation of our cubic invariant which has 15 charges. Of course, we can interpret this truncation in many different ways corresponding to the 36 different doilies residing in our  $GQ(2, 4)$ . One possibility is a truncation related to the one which employs instead of the split octonions, the split quaternions in our  $J_3(P)$ . The other is to use ordinary quaternions inside our split octonions, yielding the Jordan algebras



corresponding to the quaternionic magic. In all these cases the relevant entropy formula is related to the Pfaffian of an antisymmetric  $6 \times 6$  matrix  $A^{jk}$ ,  $i, j = 1, 2, \dots, 6$ , defined as

$$\text{Pf}(A) \equiv \frac{1}{3!2^3} \varepsilon_{ijklmn} A^{ij} A^{kl} A^{mn}. \quad (22)$$

The simplest way of finding a decomposition of  $E_{6(6)}$  directly related to a doily sitting inside  $\text{GQ}(2, 4)$  is the following one [36, 10, 37]

$$E_{6(6)} \supset SL(2) \times SL(6) \quad (23)$$

under which

$$\mathbf{27} \rightarrow (\mathbf{2}, \mathbf{6}) \oplus (\mathbf{1}, \mathbf{15}). \quad (24)$$

Clearly, this decomposition is displaying nicely its connection with the duad construction of  $\text{GQ}(2, 4)$ . One can show that under this decomposition  $I_3$  schematically factors as

$$I_3 = \text{Pf}(A) + u^T A v, \quad (25)$$

where  $u$  and  $v$  are two six-component vectors. We will have something more to say about this decomposition in the next section.

The next important type of subconfiguration of  $\text{GQ}(2, 4)$  is the grid. As we have already remarked, grids are *not* geometric hyperplanes of  $\text{GQ}(2, 4)$ . The decomposition underlying this type of subconfiguration is the one given by Eq. (6). It is also obvious that the 40 triples of pairwise disjoint grids are intimately connected to the 40 different ways we can obtain a qutrit description of  $I_3$ . Note that there are 10 grids which are geometric hyperplanes of a particular copy of the doily of  $\text{GQ}(2, 4)$ . This is related to the fact that the quaternionic magic case with 15 charges can be truncated to the complex magic case with 9 ones.

The second type of hyperplanes we should consider are perp-sets. As we already know, perp-sets are obtained by selecting an arbitrary point and considering all the points collinear with it. Since we have five lines through a point, any perp set has  $1 + 10 = 11$  points. A decomposition which corresponds to perp-sets is thus of the form [10]

$$E_{6(6)} \supset SO(5, 5) \times SO(1, 1) \quad (26)$$

under which

$$\mathbf{27} \rightarrow \mathbf{16}_1 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_4. \quad (27)$$

This is the usual decomposition of the  $U$ -duality group into the  $T$  duality and  $S$  duality [10]. It is interesting to see that the last term (i.e. the one corresponding to the fixed/central point in a perp-set) describes the  $NS$  five-brane charge. Notice that we have five lines going through this fixed point of a perp-set. These correspond to the  $T^5$  of the corresponding compactification. The two remaining points on each of these 5 lines correspond to  $2 \times 5 = 10$  charges. They correspond to the 5 directions of  $KK$  momentum and the 5 directions of fundamental string winding. In this picture the 16 charges *not belonging to* the perp-set correspond to the 16 D-brane charges. Notice that we can get 27 similar truncations based on the 27 possible central points of the perp-set. For a group theoretical meaning of the corresponding decomposition of the cubic invariant, see the paper by Borsten et al. [10].

## 4 Noncommutative coordinates for $\text{GQ}(2, 4)$

### 4.1 $\text{GQ}(2, 4)$ and qubits

The careful reader might have noticed that there is one important issue we have not clarified yet. What happened to the *signs* of the terms in the cubic invariant? Can we account for them via some sort of geometric construction?

In order to start motivating the problem of signs, we observe that the terms that should contain negative signs are the first three ones of Eq. (13), containing determinants of  $3 \times 3$

matrices. Indeed, the labelling of Figure 2 only produces the terms of the cubic invariant  $I_3$  up to a sign. One could immediately suggest that we should also include a special distribution of signs to the points of  $\text{GQ}(2, 4)$ . This would take care of the negative signs in the first three terms of Eq. (13).

However, it is easy to see that no such distribution of signs exists. The reason for this is as follows. We have a triple of grids inside our quadrangle corresponding to the three different two-qutrit states. Truncation to any of such states (say to the one with amplitudes described by the matrix  $c$ ) yields the cubic invariant  $I_3(c) = \text{Det}(c)$ . The structure of this determinant is encapsulated in the structure of the corresponding grid. We can try to arrange the 9 amplitudes in a way that the 3 plus signs for the determinant should occur along the rows and the 3 minus signs along the columns. But this is impossible since multiplying all of the nine signs “row-wise” yields a plus sign, but “column-wise” yields a minus one.

Readers familiar with the Bell-Kochen-Specker type theorems ruling out noncontextual hidden variable theories may immediately suggest that if we have failed to associate signs with the points of the grid, what about trying to use noncommutative objects instead? More precisely, we can try to associate objects that are generally noncommuting but that are pairwise commuting along the lines. This is exactly what is achieved by using Mermin squares [38, 39, 40]. Mermin squares are obtained by assigning pairwise commuting *two-qubit Pauli matrices* to the lines of the grid in such a way that the naive sign assignment does not work, but we get the identity operators with the correct signs by multiplying the operators row- and column-wise.

It is known [41] that 15 of the two-qubit Pauli operators belonging to the *two-qubit* Pauli group [15] can be associated to the points of the doily in such a way that we have mutually commuting operators along all of its 15 lines. Moreover, this assignment automatically yields Mermin squares for the 10 grids living inside the doily. Hence, a natural question to be asked is whether it is possible to use the same trick for  $\text{GQ}(2, 4)$ ? A natural extension would be to try to label the 27 points of  $\text{GQ}(2, 4)$  with a special set from the operators of the three-qubit Pauli group. In our recent paper [12] we have already gained some insight into the structure of the central quotient of this group and its connection to the  $E_7$  symmetric black hole entropy in  $D = 4$ . Hence, we can even be more ambitious and search for three-qubit labels for  $\text{GQ}(2, 4)$  also describing an embedding of our cubic invariant to the quartic one. In this way we would also obtain a new insight into the connection between the  $D = 4$  and  $D = 5$  cases in finite geometric terms.

In order to show that this program can indeed be carried out, let us define the *real* three qubit Pauli operators by introducing the notation [12]  $X \equiv \sigma_1$ ,  $Y \equiv i\sigma_2$ , and  $Z \equiv \sigma_3$ ; here,  $\sigma_j, j = 1, 2, 3$  are the usual  $2 \times 2$  Pauli matrices. Then we can define the real operators of the three-qubit Pauli group by forming the tensor products of the form  $ABC \equiv A \otimes B \otimes C$  that are  $8 \times 8$  matrices. For example, we have

$$ZYX \equiv Z \otimes Y \otimes X = \begin{pmatrix} Y \otimes X & 0 \\ 0 & -Y \otimes X \end{pmatrix} = \begin{pmatrix} 0 & X & 0 & 0 \\ -X & 0 & 0 & 0 \\ 0 & 0 & 0 & -X \\ 0 & 0 & X & 0 \end{pmatrix}. \quad (28)$$

Notice that operators containing an even number of  $Y$ s are *symmetric* and the ones containing an odd number of  $Y$ s are *antisymmetric*. Disregarding the identity,  $III$ , ( $I$  is the  $2 \times 2$  identity matrix) we have 63 of such operators. We have shown [12] that they can be mapped bijectively to the 63 points of the split Cayley hexagon of order two in such a way that its 63 lines are formed by three pairwise commuting operators. These 63 triples of operators have the property that their product equals  $III$  up to a sign.

It is easy to check that the 35 symmetric operators form a geometric hyperplane of the hexagon. Its complement is the famous Coxeter graph, whose vertices are labelled by the 28 antisymmetric matrices. It was shown [12] that the automorphism group of both of these subconfigurations is  $PSL(2, 7)$ , having a generator of order seven. Due to this we can group the 28 antisymmetric operators to 4 seven-element sets. One of these sets is

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7) = (IIY, ZYX, YIX, YZZ, XYX, IYZ, YXZ) \quad (29)$$

satisfying the relation  $\{g_a, g_b\} = -2\delta_{ab}$ ,  $a, b = 1, 2, \dots, 7$ , i.e. these operators form the generators of a seven-dimensional Clifford algebra. Notice that these generators, up to some sign conventions and a cyclic permutation, are precisely the ones used by Cremmer and Julia in their classical paper [42] on  $SO(8)$  supergravity. Namely, their generators  $\gamma^a$ ,  $a = 4, 5, \dots, 10$  have the form

$$\{\gamma^4, \gamma^5, \gamma^6, \gamma^7, \gamma^8, \gamma^9, \gamma^{10}\} = \{ZYX, -YZZ, ZIY, XXY, XYI, -XZY, -YII\}. \quad (30)$$

The remaining 21 antisymmetric operators are of the form  $\frac{1}{2}[g_a, g_b]$ ,  $a, b = 1, 2, \dots, 7$ , i.e. they generate an  $so(7)$  algebra. One can then form the  $8 \times 8$  matrix  $-\Gamma^{IJ}$ ,  $I, J = 0, 1, \dots, 7$  whose entries are our 28 antisymmetric matrices

$$-\Gamma^{0a} = g_{0a} \equiv g_a, \quad -\Gamma^{ab} = g_{ab} \equiv \frac{1}{2}[g_a, g_b]. \quad (31)$$

In other words,  $(\Gamma^{IJ})_{AB}$ ,  $A, B = 0, 1, \dots, 7$  are generators of the  $so(8)$  algebra in the spinor representation. Hence, we managed to relate the 28 generators of  $so(8)$  to the complement of one of the geometric hyperplanes of the split Cayley hexagon of order two, namely to the Coxeter graph.

Notice that Eqs. (29) and (31) give an explicit labelling for the 28 points of the Coxeter graph in terms of three-qubit operators. We can make use of this structure by employing these three-qubit operators for expanding the  $N = 8$  central charge  $\mathcal{Z}_{AB}$  as

$$\mathcal{Z}_{AB} = -(x^{IJ} + iy_{IJ})(\Gamma^{IJ})_{AB}, \quad (32)$$

where summation for  $I < J$  is implied and the real antisymmetric matrices  $x^{IJ}$  and  $y_{IJ}$  describe the 28 electric and 28 magnetic charges which are related to some numbers of membranes wrapping around the extra dimensions where these objects live in [45].

In order to establish a connection between the  $D = 4$  and  $D = 5$  cases, we assign to one of the 28 antisymmetric three-qubit operators a special status. Later, we will show that this choice amounts to a choice of the symplectic structure  $\Omega$  in the usual formalism of  $D = 5$  black hole solutions [24]. Let us make the following choice

$$\Omega = IYY = g_1 = g_{01} = -\Gamma^{01}. \quad (33)$$

(The usual choice for  $\Omega$  is  $YII$  [42, 24].) Then recalling the duad construction of  $GQ(2, 4)$ , a natural choice to try for the labelling of the 27 points of our quadrangle is

$$\{1', 2', 3', 4', 5', 6'\} \leftrightarrow \{g_{02}, g_{03}, g_{04}, g_{05}, g_{06}, g_{07}\} = \{g_2, g_3, g_4, g_5, g_6, g_7\}, \quad (34)$$

$$\{1, 2, 3, 4, 5, 6\} \leftrightarrow \{g_{12}, g_{13}, g_{14}, g_{15}, g_{16}, g_{17}\}, \quad (35)$$

$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\} \leftrightarrow \{g_{23}, g_{24}, g_{25}, g_{26}, g_{27}, g_{34}, g_{35}, g_{36}, g_{37}\}, \quad (36)$$

$$\{34, 35, 36, 45, 46, 56\} \leftrightarrow \{g_{45}, g_{46}, g_{47}, g_{56}, g_{57}, g_{67}\}, \quad (37)$$

i.e., shifting all the indices of  $g_{IJ}$  not containing 0 or 1 by  $-1$  we get the duad labels.

Now using the explicit form of the antisymmetric operators  $g_{ab}$ ,  $a, b \neq 0, 1$  used to label the points of our generalized quadrangle we notice that for all of the 45 lines the product of the corresponding 3 three-qubit operators gives, up to a sign,  $\Omega$ ! Moreover, we also realize that the 15 triples of operators associated with the 15 lines of the doily are pairwise commuting. However, the triples of operators belonging to the 30 lines featuring the double-sixes outside the doily fail to be pairwise commuting. But we also notice that for such lines the 2 operators belonging to the double-sixes are always commuting, but either of them anticommutes with the remaining operator belonging to the doily. It is also clear that  $\Omega$  *anticommutes* with the operators of the double-sixes, and *commutes* with the ones of the doily. Hence, if we multiply all the operators belonging to the doily by  $\Omega$ , the resulting *symmetric* ones will preserve the nice pairwise commuting property, and at the same time

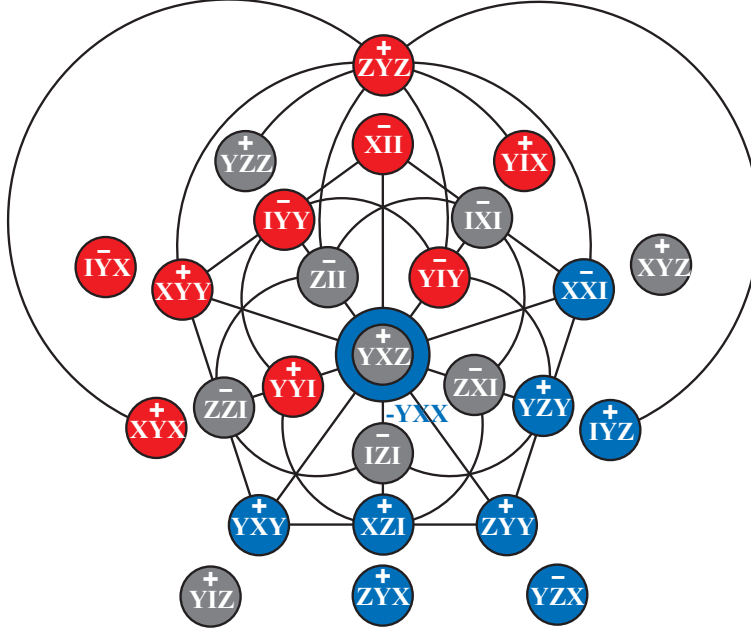


Figure 3: An illustration of the non-commutative labelling of the points of  $GQ(2,4)$ . For better readability of the figure, the sign of an operator is placed above the latter.

the same property is also achieved for the resulting *antisymmetric* operators featuring the lines of the double sixes. And as an extra bonus: the product of all triples of operators along the lines gives  $III$ , again up to a sign. In this way we have obtained a sort of *non-commutative* labelling for the points of  $GQ(2,4)$ . The 15 points of the doily are labelled by 15 symmetric operators, and the 12 double-sixes are labelled by antisymmetric ones. The incidence relation on this set of 27 points producing the 45 lines is: a pairwise commuting property and a “sum rule” (i.e. multiplication producing  $III$  up to a sign).

Notice that for  $a, b, c = 1, 2, \dots, 7$  the combinations  $g_{abc} \equiv g_a g_b g_c$  as elements of the Clifford algebra  $Cliff(7)$  are symmetric and the ones  $g_a$  and  $g_{ab} = g_a g_b$  are antisymmetric matrices with some signs automatically incorporated. Hence, the simplest choice for a labelling taking care of the signs is simply

$$\{1', 2', 3', 4', 5', 6'\} = \{g_2, g_3, g_4, g_5, g_6, g_7\}, \quad (38)$$

$$\{1, 2, 3, 4, 5, 6\} = \{g_{12}, g_{13}, g_{14}, g_{15}, g_{16}, g_{17}\}, \quad (39)$$

$$\{12, 13, 14, 15, 16, 23, 24, 25, 26\} = \{g_{123}, g_{124}, g_{125}, g_{126}, g_{127}, g_{134}, g_{135}, g_{136}, g_{137}\}, \quad (40)$$

$$\{34, 35, 36, 45, 46, 56\} = \{g_{145}, g_{146}, g_{147}, g_{156}, g_{157}, g_{167}\}. \quad (41)$$

Using the explicit form of the  $8 \times 8$  matrices  $g_a$ ,  $a = 1, 2, \dots, 7$  of Eq. (29), we can get three-qubit operators with a natural choice of signs as non-commutative labels for the points of  $GQ(2,4)$ . The summary of this chain of reasoning is displayed in Figure 3.

## 4.2 The Weyl action on $GQ(2,4)$

Using our new labelling we can demonstrate the  $W(E_6)$  invariance of  $GQ(2,4)$ . This renders our arguments on the relationship between the structure of  $GQ(2,4)$  and  $I_3$  to a proof.

Let us consider the correspondence

$$I \mapsto (00), \quad X \mapsto (01), \quad Y \mapsto (11), \quad Z \mapsto (10). \quad (42)$$

Using this we can map an arbitrary element of the central quotient of the three-qubit Pauli group to  $\mathbf{Z}_2^6$ , i. e. to the space of 6-component vectors with elements taken from  $GF(2)$ . For example,  $XZI$  is taken to the 6-component vector (011000). Clearly, if we are interested merely in the incidence structure then we can label the points of  $GQ(2, 4)$  with such six component vectors. Knowing that  $W(E_6) \cong O^-(6, 2)$ , which is the set of  $6 \times 6$  matrices with entries taken from  $GF(2)$  leaving invariant a special quadratic form [43] defined on  $\mathbf{Z}_2^6$ , we can check the Weyl invariance by checking the invariance under a suitable set of generators. From the atlas of finite groups [44] we use the presentation

$$O^-(6, 2) = U(4, 2) \rtimes \mathbf{Z}_2 = \langle c, d | c^2 = d^9 = (cd^2)^8 = [c, d^2]^2 = [c, d^3cd^3] = 1 \rangle. \quad (43)$$

We have found the following representation convenient (this is the one that is preserving the symplectic structure corresponding to the commutation properties in the Pauli group [15], and mapping the 27 three-qubit label onto itself)

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}. \quad (44)$$

Using the above-given dictionary, we explicitly get for the action of  $c$

$$IXI \mapsto XZI, \quad ZYX \mapsto YIX, \quad IZI \mapsto XXI \quad (45)$$

$$ZYZ \mapsto YIZ, \quad ZII \mapsto YYI, \quad ZYY \mapsto YIY, \quad (46)$$

and the remaining 15 operators are left invariant. For the action of  $d$  we get three orbits

$$IXI \mapsto YXZ \mapsto YZX \mapsto YIX \mapsto XYZ \mapsto IYZ \mapsto YXX \mapsto ZZI \mapsto YXY \quad (47)$$

$$IZI \mapsto ZYY \mapsto XII \mapsto YZY \mapsto XYX \mapsto XYY \mapsto YIY \mapsto YIZ \mapsto IYY \quad (48)$$

$$IYX \mapsto ZXI \mapsto ZYZ \mapsto ZYX \mapsto YYI \mapsto YZZ \mapsto ZII \mapsto XZI \mapsto XXI. \quad (49)$$

One can check that these generators take lines to lines, hence preserving  $GQ(2, 4)$ .

Moreover, using this action of  $W(E_6)$  on  $GQ(2, 4)$  we can define a corresponding one on  $GQ(2, 4)$  taken together with the non-commutative coordinates. For this we can take the very same expressions as above but taking also into account the *signs* of the operators, as shown in Figure 3. Since these signed quantities automatically take care of the structure of signs of  $I_3$ , this furnishes a proof for the  $W(E_6)$  invariance of  $I_3$ . Notice also that the transformation rules for the non-commutative labels imply the corresponding rule for the *charges*. Having in this way an explicit action on the charges and the invariance of the black hole entropy, it would be interesting to work out manifestations of this discrete symmetry of order 51840 in string theory.

### 4.3 A $D = 4$ interpretation

Note that the decomposition

$$E_{7(7)} \supset E_{6(6)} \times SO(1, 1) \quad (50)$$

under which

$$\mathbf{56} \rightarrow \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{27}' \oplus \mathbf{1}' \quad (51)$$

describes the relation between the  $D = 4$  and  $D = 5$  duality groups [46, 47, 48]. We intend to show that the non-commutative labelling constructed for our quadrangle provides a nice finite geometric interpretation of the physics based on the decomposition of Eq. (50).

To this end, we use the  $N = 8$  central charge parametrized as in Eq. (32) and look at the structure of the cubic invariant that can be written also in the alternative form [45]

$$I_3 = \frac{1}{48} \text{Tr}(\Omega \mathcal{Z} \Omega \mathcal{Z} \Omega \mathcal{Z}) \quad (52)$$

where for  $\Omega$  we use the definition of Eq. (33). In order to get the correct number of components, we impose the usual constraints [24]

$$\text{Tr}(\Omega \mathcal{Z}) = 0, \quad \overline{\mathcal{Z}} = \Omega \mathcal{Z} \Omega^T. \quad (53)$$

Notice that the first of these constraints restricts the number of antisymmetric matrices to be considered in the expansion of  $\mathcal{Z}$  from 28 to 27. The second constraint is the usual reality condition which restricts the 27 *complex* expansion coefficients to 27 *real* ones, producing the right count. Recall also that the group theoretical meaning of these constraints is the expansion of the  $N = 8$  central charge in an  $USp(8)$  basis, which is appropriate since  $USp(8)$  is the automorphism group of the  $N = 8$ ,  $D = 5$  supersymmetry algebra.

It is easy to see that the reality constraint yields

$$y_{jk} = 0, \quad x^{0j} = 0, \quad x^{1j} = 0, \quad j, k = 2, 3, \dots, 7, \quad (54)$$

hence  $\Omega \mathcal{Z}$  is of the form

$$\Omega \mathcal{Z} = \mathcal{S} + i\mathcal{A} \equiv \frac{1}{2} x^{jk} g_{1jk} + i(y_{0j} g_{1j} - y_{1j} g_j), \quad (55)$$

where summation for  $j, k = 2, 3, \dots, 7$  is understood. The new notation for  $\Omega \mathcal{Z}$  shows that  $\mathcal{S}$  is *symmetric* and  $\mathcal{A}$  is *antisymmetric*. Notice that the three-qubit operators occurring in the expansions of  $\mathcal{S}$  and  $\mathcal{A}$  are precisely the ones we used in Eqs. (38)–(41) as our non-commutative “coordinates” for  $\text{GQ}(2, 4)$ .

Performing standard manipulations, we get

$$I_3 = \frac{1}{48} (\text{Tr}(\mathcal{S}\mathcal{S}\mathcal{S}) - 3\text{Tr}(\mathcal{S}\mathcal{A}\mathcal{A})). \quad (56)$$

Notice that

$$\Omega g_{i+1} g_{j+1} g_{k+1} g_{l+1} g_{m+1} g_{n+1} = -\varepsilon_{ijklmn}, \quad i, j, k, l, m, n = 1, 2, \dots, 6, \quad \varepsilon_{123456} = +1. \quad (57)$$

Hence, with the notation

$$A^{jk} \equiv x^{j+1k+1}, \quad u_j \equiv y_{0j+1}, \quad v_j \equiv y_{1j+1}, \quad j, k = 1, 2, \dots, 6, \quad (58)$$

the terms of Eq. (56) give rise to the form of Eq. (25). Also notice that the parametrization

$$u^T = (b^{10}, \quad -c_{10}, \quad a^1{}_2, \quad c_{12}, \quad b^{12}, \quad a^1{}_0), \quad (59)$$

$$v^T = (-c_{21}, \quad -a^2{}_1, \quad -b^{01}, \quad -a^0{}_1, \quad c_{01}, \quad b^{21}), \quad (60)$$

$$A = \begin{pmatrix} 0 & c_{02} & b^{22} & -c_{00} & a^1{}_1 & b^{02} \\ -c_{02} & 0 & a^0{}_0 & b^{11} & c_{22} & -a^0{}_2 \\ -b^{22} & -a^0{}_0 & 0 & a^2{}_0 & b^{20} & c_{11} \\ c_{00} & -b^{11} & -a^2{}_0 & 0 & c_{20} & a^2{}_2 \\ -a^1{}_1 & -c_{22} & -b^{20} & -c_{20} & 0 & -b^{00} \\ -b^{02} & a^0{}_2 & -c_{11} & -a^2{}_2 & b^{00} & 0 \end{pmatrix}, \quad (61)$$

yields the qutrit version of  $I_3$  of Eq. (13).

The main message of these considerations is obvious: different versions of  $I_3$ , and, so, of the black hole entropy formula, are obtained as different parametrizations of the underlying finite geometric object — our generalized quadrangle  $\text{GQ}(2, 4)$ .

## 4.4 Mermin squares and the hyperplanes of the hexagon

Our non-commutative coordinatization of  $\text{GQ}(2, 4)$  in terms of the elements of  $\text{Cliff}(7)$ , or equivalently by three-qubit operators, is very instructive. For example, one can easily check that this labelling for the doily gives rise to 7 lines with a minus sign and 8 lines with a plus one. (That is, the product of the corresponding operators yields either  $-III$  or  $+III$ .) This is in accord with the sign structure of the Pfaffian. It is easy to check that for each of the 10 grids living inside the doily these signs give rise to 3 plus signs and 3 minus ones needed for producing the determinant related to the 10 possible truncations with 9 charges. These 10 grids generate 10 Mermin squares.

As repeatedly mentioned, inside  $\text{GQ}(2, 4)$  there are also triads of grids which are partitioning its 27 points. These are the ones related to the three qutrit states, indicated by coloring the corresponding points in three different ways. They are also producing Mermin squares. Note, the usual definition of a Mermin square is a grid having the property that the products of operators along any of its rows and columns *except for one* yield  $III$ . Here, we define Mermin squares as objects for which no simple sign assignment can produce the rule the operator products give. In this generalized sense we have  $3 \times 40 = 120$  Mermin squares living inside our  $\text{GQ}(2, 4)$ .

Of course, our particular “coordinates” producing the three special Mermin-squares can be replaced by other possible ones arising from 27 further labellings. In order to see this notice that the “non-commutative coordinates” of Figure 3 are the ones based on a special choice for the matrix  $\Omega$  of Eq. (33). Since we have 28 antisymmetric operators, we have 27 further possible choices for  $\Omega$ . Choosing *any* of these matrices will produce a  $27 = 12 + 15$  split for the space of the remaining antisymmetric operators. For this we simply have to consider the 12 operators *anticommuting* and the 15 ones *commuting* with our fixed  $\Omega$ . This can be easily checked using the property that the antisymmetric matrices are either of the form  $g_{ab}$  or  $g_a, a = 1, 2, \dots, 7$ . Now apply the simple rule: multiply the 15 operators commuting with  $\Omega$  by  $\Omega$  and leave the remaining ones untouched. One can then check that this procedure will yield 27 further possible non-commutative labels for the points of  $\text{GQ}(2, 4)$ , hence another possible sets of Mermin squares. Notice also that for the special choice  $\Omega = YYY$  the reality condition of Eq. (53) is related to the three-qubit version of the so-called Wootters spin-flip operation [50] used in quantum information.

We round off this section with an important observation. The Coxeter set comprising the points of the generalized hexagon of order two answering to the set of antisymmetric three-qubit operators is just the complement of one of the geometric hyperplanes of the hexagon. The 28 possibilities for fixing  $\Omega$  gives rise to 28 subconfigurations consisting of 27 points. These 27 points are always consisting of 12 antisymmetric operators and 15 symmetric ones. By picturing them inside the hexagon [12], one can realize that any such subconfiguration consists of 9 pairwise disjoint lines (i.e., is a distance-3-spread). It turns out that these subconfigurations are also *geometric hyperplanes* living inside the hexagon [49]. Hence, we have found a very interesting geometric link between the structures of  $D = 4$  and  $D = 5$  entropy formulas. The  $D = 4$  case is related to the split Cayley hexagon of order two [12] and here we have demonstrated that the  $D = 5$  one is underlined by the geometry of the generalized quadrangle  $\text{GQ}(2, 4)$ . The connection between these cases is based on a beautiful relationship between the structure of  $\text{GQ}(2, 4)$  and one of the geometric hyperplanes of the hexagon.

## 5 Conclusion

In this paper we revealed an intimate connection between the structure of black hole entropy formulas in  $D = 4$  and  $D = 5$  and the geometry of certain finite generalized polygons. We provided a detailed correspondence between the structure of the cubic invariant related to the black hole entropy in  $D = 5$  and the geometry of the generalized quadrangle  $\text{GQ}(2, 4)$  with automorphism group the Weyl group  $W(E_6)$ . In this picture the 27 charges correspond to the points and the 45 terms in the entropy formula to the lines of  $\text{GQ}(2, 4)$ . Different truncations with 15, 11 and 9 charges are represented by three distinguished subconfigurations of  $\text{GQ}(2, 4)$ , well-known to finite geometers; these are the “doily” (i.e.  $\text{GQ}(2, 2)$ ) with 15, the

“perp-set” of a point with 11, and the “grid” (i. e.  $\text{GQ}(2, 1)$ ) with 9 points, respectively. Different truncations naturally employ objects like cubic Jordan algebras well-known to string theorists, or qubits and qutrits well-known to quantum information theorists. In our finite geometric treatment these objects just provide different coordinates for the underlying geometric object,  $\text{GQ}(2, 4)$ . However, in order to account also for the signs of the monomials in the cubic invariant, the labels, or “coordinates” used for the points of  $\text{GQ}(2, 4)$  must be non-commutative. We have shown that the real operators of the three-qubit Pauli group provide a natural set of such coordinates. An alternative way of looking at these coordinates is obtained by employing a special 27 element set of  $\text{Cliff}(7)$ . Hence it seems quite natural to conjecture that the different possibilities of describing the  $D = 5$  entropy formula using Jordan algebras, qubits and/or qutrits merely correspond to employing different coordinates for an underlying noncommutative geometric structure based on  $\text{GQ}(2, 4)$ .

Using these coordinates we established the Weyl invariance of the cubic invariant and we also shed some light on the interesting connection between the different possible truncations with 9 charges and the geometry of Mermin squares — objects well-known from studies concerning Bell-Kochen-Specker like theorems. Since these 9-charge configurations as qutrits can also be connected to special brane configurations [8], it would also be nice to relate their physical properties to these Mermin squares.

We emphasize that these results are also connected to our previous ones obtained for the  $E_7$  symmetric entropy formula in  $D = 4$  by observing that the structure of  $\text{GQ}(2, 4)$  is linked to a particular geometric hyperplane of the split Cayley hexagon of order two [12] featuring 27 points located on 9 pairwise disjoint lines (a distance-3-spread). This observation provides a direct finite geometric link between the  $D = 4$  and  $D = 5$  cases. However, there are other interesting hyperplanes of the hexagon. Their physical meaning (if any) is not clear. In particular, we have other three distinct types of hyperplanes with 27 points inside the hexagon [49]. They might shed some light on the geometry of further truncations that are not arising so naturally as the ones discussed in this paper.

Finally, it is worth mentioning that the above-employed generalized quadrangles with lines of size three are also closely related with particular root lattices [20]. Given an irreducible root lattice  $\Lambda$ , one picks any two roots  $a, b$  whose inner product equals unity,  $\langle a, b \rangle = 1$  (whence  $\langle a, a \rangle = \langle b, b \rangle = 2$ ). Then the set  $S = \{r \in \Lambda | \langle r, r \rangle = 2, \langle r, a \rangle = \langle r, b \rangle = 1\}$  is a generalized quadrangle with lines of size three if the latter are represented by the triples  $\{x, y, z\}$  meeting the constraint  $x + y + z = a + b$ . Since  $\Lambda$  is spanned by  $\{a, b\} \cup S$ , the structure of  $S$  determines  $\Lambda$ . And it turns out [20] that the root lattices that correspond to  $\text{GQ}(2, 1)$ ,  $\text{GQ}(2, 2)$ , and  $\text{GQ}(2, 4)$ , are nothing but those of  $E_6$ ,  $E_7$  and  $E_8$ , respectively.

## Acknowledgement

This work was partially supported by the VEGA grant agency projects Nos. 2/0092/09 and 2/7012/27.

## References

- [1] M. J. Duff, Phys. Rev. D**76**, 025017 (2007).
- [2] R. Kallosh and A. Linde, Phys. Rev. D**73**, 104033 (2006).
- [3] P. Lévay, Phys. Rev. D**74**, 024030 (2006).
- [4] S. Ferrara and M. J. Duff, Phys. Rev. D**76**, 025018 (2007).
- [5] P. Lévay, Phys. Rev. D**75**, 024024 (2007).
- [6] P. Lévay, Phys. Rev. D**76**, 106011 (2007).
- [7] S. Ferrara and M. J. Duff, Phys. Rev. D**76**, 124023 (2007).



- [8] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, and W. Rubens, Phys. Rev. Lett. **100**, 251602 (2008).
- [9] L. Borsten, Fortschr. Phys. **56**, 842 (2008).
- [10] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, and W. Rubens, arXiv:0809.4685.
- [11] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, and W. Rubens, arXiv:0812.3322.
- [12] P. Lévy, M. Saniga and P. Vrana, Phys. Rev. D **78**, 124022 (2008).
- [13] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
- [14] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
- [15] M. A. Nielsen and I. L. Chuang, *Quantum Information and Quantum Computation*, Cambridge University Press, Cambridge, 2000.
- [16] W. Feit and D. G. Higman, Journal of Algebra **1**, 114 (1964).
- [17] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
- [18] R. C. Bose and S. S. Shrikhande, Journal of Geometry **2**, 75 (1972).
- [19] D. G. Higman, Partial geometries, generalized quadrangles and strongly regular graphs, Atti del Convegno di Geometria Combinatoria e sue Applicazioni (Univ. degli Studi di Perugia), Perugia, 1971, pp. 263–293.
- [20] A. E. Brouwer and W. H. Haemers, Spectra of Graphs (Chapter 8.6.4), unpublished course notes available at <http://homepages.cwi.nl/~aeb/math/ipm.pdf>.
- [21] M. Gunaydin, G. Sierra, and P. K. Townsend, Nucl. Phys. B **242**, 244 (1984).
- [22] M. Gunaydin, G. Sierra, and P. K. Townsend, Phys. Lett. B **133**, 72 (1983).
- [23] M. Gunaydin, G. Sierra, and P. K. Townsend, Nucl. Phys. B **253**, 573 (1985).
- [24] S. Ferrara, E. G. Gimon and R. Kallosh, Phys. Rev. D **74**, 125018 (2006).
- [25] S. Ferrara and J. M. Maldacena, Class. Quant. Grav. **15**, 749 (1998).
- [26] S. Ferrara and R. Kallosh, Phys. Rev. D **54**, 1525 (1996).
- [27] L. Andrianopoli, R. D’Auria, and S. Ferrara, Phys. Lett. B **411**, 39 (1997).
- [28] S. Ferrara and M. Gunaydin, Int. J. Mod. Phys. A **13**, 2075 (1998).
- [29] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, Pitman, Boston–London–Melbourne, 1984; see also K. Thas, *Symmetry in Finite Generalized Quadrangles*, Birkhäuser, Basel, 2004.
- [30] M. A. Ronan, European J. Combin. **8**, 179 (1987).
- [31] B. Polster, *A Geometrical Picture Book*, Springer, New York, 1991, pp. 59–60.
- [32] B. Pioline, Class. Quant. Grav. **23**, S981 (2006).
- [33] L. Manivel, Journal of Algebra **304**, 457 (2006).
- [34] E. Cartan, Amer. J. Math. **18**(1), 1 (1986).
- [35] M. Saniga, P. Lévy, P. Pracna and P. Vrana, “The Veldkamp Space of GQ(2, 4),” arXiv:0903.0715.
- [36] L. Andrianopoli, R. D’Auria, and S. Ferrara, Nucl. Phys. Proc. Suppl. **67**, 17 (1998).
- [37] J. C. Baez, Bull. Amer. Math. Soc. **39**, 145 (2002).

- [38] N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
- [39] N. D. Mermin, Rev. Mod. Phys. **65**, 803 (1993).
- [40] A. Peres, J. Phys. A: Math. Gen. **24**, L715 (1991).
- [41] M. Saniga, M. Planat, P. Pracna and H. Havlicek, SIGMA **3**, 075 (2007); M. Saniga, M. Planat and P. Pracna, Theor. and Math. Phys. **155**, 905 (2008).
- [42] E. Cremmer and B. Julia, Nucl. Phys. B**159**, 141 (1979).
- [43] See R. Shaw, “Finite geometry, Dirac groups and the table of real Clifford algebras,” in R. Abłamowicz and P. Lounesto (eds.), Clifford Algebras and Spinor Structures, Kluwer Academic Publishers, Dordrecht, 1995, pp. 59–99.
- [44] J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson, *An Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.
- [45] K. Becker, M. Becker and J. H. Schwartz, *String Theory and M-Theory. A Modern Introduction*, Cambridge University Press, Cambridge, 2007.
- [46] S. Ferrara and R. Kallosh, Phys. Rev. D**54**, 1525 (1996).
- [47] R. Kallosh and B. Kol, Phys. Rev. D**53**, R5344 (1996); M. Cvetič and C. M. Hull, Nucl. Phys. B**480**, 296 (1996); V. Balasubramanian, F. Larsen, and R. G. Leigh, Phys. Rev. D**57**, 3509 (1998); M. Bertolini and M. Trigiante, Nucl. Phys. B**582**, 393 (2000).
- [48] L. Andrianopoli, R. D’Auria, S. Ferrara, P. Fre, and M. Trigiante, Nucl. Phys. B**509**, 463 (1998).
- [49] D. Frohardt and P. Johnson, Communications in Algebra **22**, 773 (1994).
- [50] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).